Poisson Distribution

IE231 - Lecture Notes 7

Mar 28, 2017

Poisson distribution is widely used to represent occurences in an interval, mostly time but sometimes area. Examples include arrivals to queues in a day, number of breakdowns in a machine in a year, typos in a letter, oil reserve in a region.

Binomial Approximation to Poisson Distribution

We know from binomial distribution that k occurences in n trials with probability p has the following function.

$$P\{X=k\} = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k}$$

and expected value is $E[X] = np$. Now define $\lambda = np$.

$$\begin{aligned} P\{X=k\} =& \frac{n!}{(n-k)!k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \left(\frac{\lambda}{n}\right)\right)^{n-k} \\ =& \frac{n(n-1)\dots(n-k+1)}{n^k} \left(\frac{\lambda^k}{k!}\right) \frac{(1-\frac{\lambda}{n})^n}{(1-\frac{\lambda}{n})^k} \end{aligned}$$

For very large n and very small p the resulting pdf becomes $\dfrac{\lambda^k e^{-\lambda}}{k!}$.

Properties of Poisson Distribution

- PMF: $P\{X=k\} = \dfrac{\lambda^k e^{-\lambda}}{k!}$
- CDF: $P\{X \leq k\} = \sum_{i=0}^{k} \dfrac{\lambda^i e^{-\lambda}}{i!}$
- $E[X] = \lambda (due to \sum_{i=0}^{\infty} \dfrac{x^i}{i!} = e^x)$
- $V(X) = \lambda$

Rate parameter λ can also be defined as λt, t being the scale parameter. For instance, let arrivals in 30 minutes interval be $\lambda t_{30} = 4$. If we would want to work on hourly intervals, we should simply rescale, $\lambda t_{60} = 8$.

Examples

Example 1

Suppose a machine has a probability of failure 0.001 per hour. What is the probability that the machine had failed at least three times within 2000 hours.

Binomial solution

$$\begin{aligned}
P\{X \geq 3\} &= 1 - \binom{2000}{0} 0.001^0 0.999^{750} - \binom{2000}{1} 0.001^1 0.999^{749} - \binom{2000}{2} 0.001^2 0.999^{748} \\
&= 0.3233236
\end{aligned}$$

```r
#R version
1- sum(dbinom(0:2,2000,0.001))
```

```
## [1] 0.3233236
```

Poisson solution

$$\begin{aligned}
\lambda &= np = 2000 * 0.001 = 2 \\
P\{X \geq 3\} &= 1 - \frac{e^{-2} 2^0}{0!} - \frac{e^{-2} 2^1}{1!} - \frac{e^{-2} 2^2}{2!} \\
&= 0.3233236
\end{aligned}$$

```r
#R version
lambda=2000*0.001
1- sum(dpois(0:2,lambda))
```

```
## [1] 0.3233236
```

Example 2

People arrive at a bank with rate $\lambda = 5$ every 10 minutes. What is the probability that 10 people arrive in 30 minutes?

$$\lambda t_{10} = 5$$

$$\lambda' = \lambda t_{30} = 15$$

$$P\{X = 10, t = 30\} = \frac{e^{-15} 15^{10}}{10!} = 0.049$$

```r
dpois(10,15)
```

```
## [1] 0.04861075
```

Example 3

A machine breaks down with a poisson rate of $\lambda = 10$ per year. A new method is tried to reduce the failure rate to $\lambda = 3$, but there is a 50% chance that it won't work. If the method is tried and the machine fails only 3 times that year, what is the probability that the method worked on the machine?

$$P\{Works|X=3\} = \frac{P\{WorksandX=3\}}{P\{X=3\}}$$

$$P\{Works\} = 0.5$$

$$P\{WorksandX=3\} = 0.5 * \frac{e^{-3}3^3}{3!} = 0.1120209$$

$$\begin{aligned} P\{X=3\} &= P\{WorksandX=3\} + P\{Doesn't\ WorkandX=3\} = 0.5 * \frac{e^{-3}3^3}{3!} + 0.5 * \frac{e^{-10}10^3}{3!} \\ &= 0.96733 \end{aligned}$$

```
#R codes
#Probability that it works
pw = 0.5
#Probability of 3 fails if lambda is 10
ppois10 = dpois(3,10)
#Probability of 3 fails if lambda is 3
ppois3 = dpois(3,3)
#P(Works | X = 3)
(pw*ppois3)/(pw*ppois3 + (1-pw)*ppois10)
```

```
## [1] 0.96733
```