Some Continuous Distributions

IE231 - Lecture Notes 9

Mar 28, 2017

So far we had only seen discrete distributions which only specific values have positive probability values. Now we are going to see continuous variables where each real value defined in the domain of the distribution has a positive probability.

In this class we are going to see uniform, exponential, normal, gamma and weibull distributions. Of those, we will see uniform, exponential and normal distributions in detail.
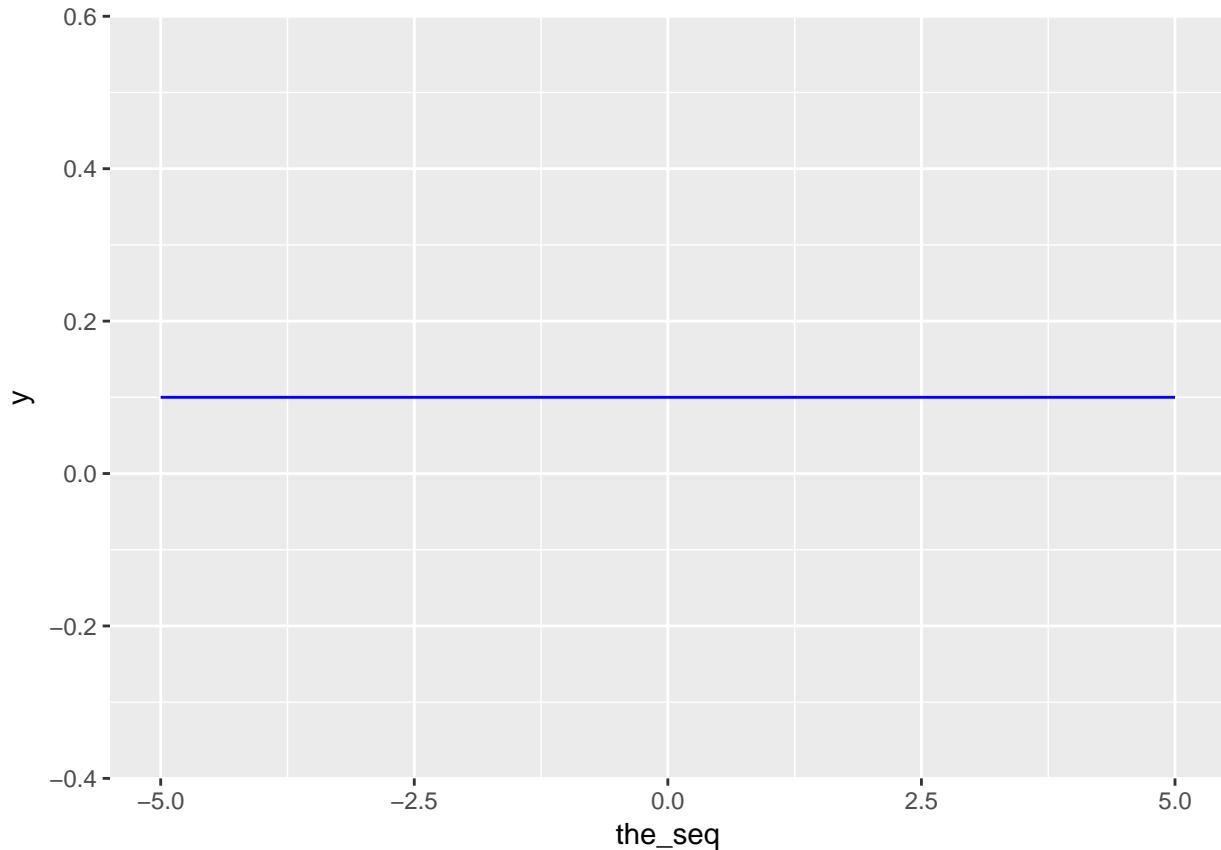
Uniform Distribution

Given an interval $[a, b]$, each value within the interval has equal probability in uniform distribution.

$$X \sim U[a, b]$$

- Density: $f(X) = \dfrac{1}{b - a}$

- CDF: $F(X \leq x) = \dfrac{x - a}{b - a}$

- $E[X] = (b + a)/2$

- $V(X) = 1/12(b - a)^2$

```
the_seq <- seq(-5,5,length.out=1000)

ggplot(data=data.frame(the_seq),aes(x=the_seq)) + stat_function(fun=dunif,args=list(min=-5,max=5),color=
```

Example: Suppose there is a lecture that can end anytime between 11:00 and 13:00. What is the probability that it ends before 12:30?

Solution: There are 120 minutes betweeen 11:00 (a) and 13:00 (b) and 90 minutes between 11:00 (a) and 12:30 (x).

$$P(X \leq 90) = 90/120 = 3/4$$

Exponential Distribution

Exponential distribution is generally used to measure time before an event happens. Common examples are component (i.e. light bulb) lifetime and job processing (i.e. queue serving). It is closely related to Poisson distribution. While Poisson is used to estimate number of events in a given time period, Exponential distribution estimates the time of an event.

- Density: $f(X) = \lambda e^{-\lambda x}$
- CDF: $F(X \leq x) = 1 - e^{-\lambda x}$
- $E[X] = 1/\lambda$
- $V(X) = 1/\lambda^2$
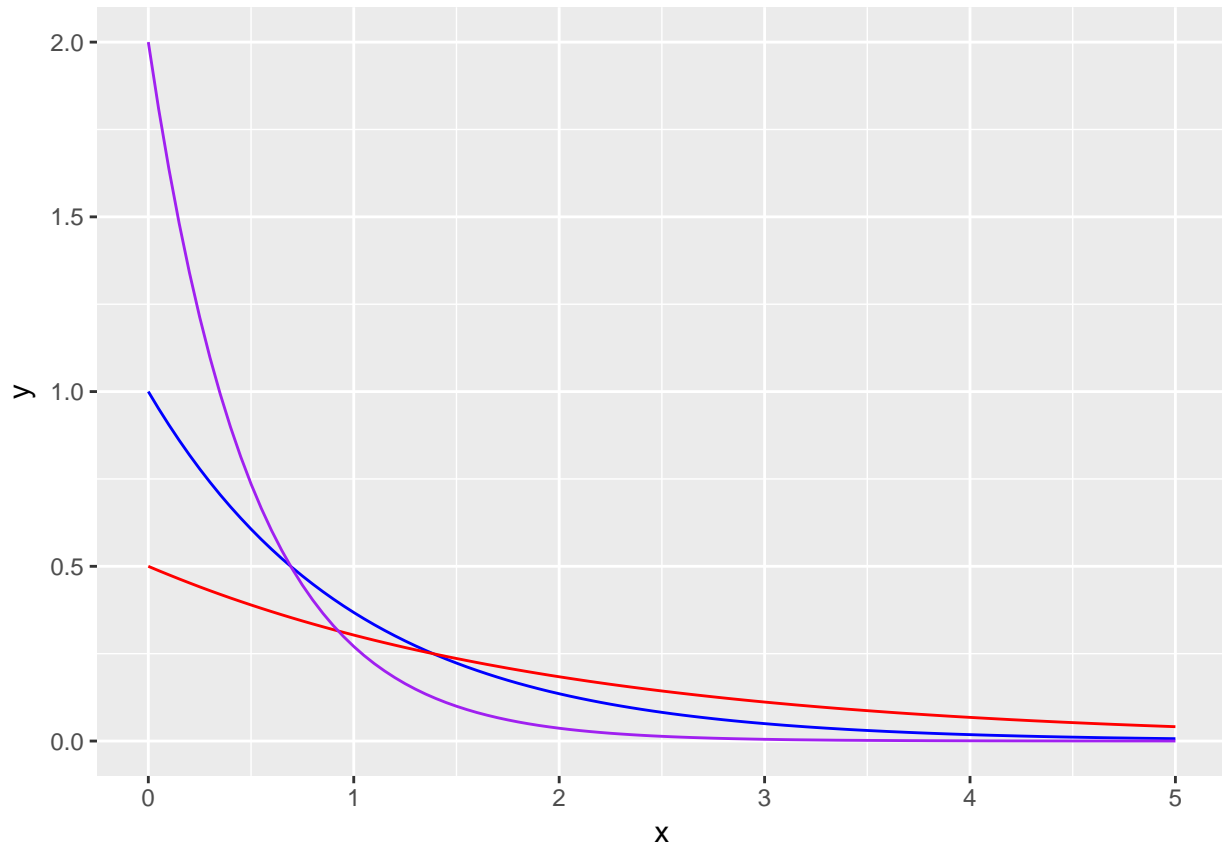
Exponential distribution has memoryless property.

$$P(X > t + s | X > t) = \frac{P(X > t + s)}{P(X > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s}$$

```
the_seq <- seq(0,5,length.out=1000)

ggplot(data=data.frame(x=the_seq),aes(x=x)) +
stat_function(fun=dexp,color="blue") +
stat_function(fun=dexp,args=list(rate=0.5),color="red") +
stat_function(fun=dexp,args=list(rate=2),color="purple")
```


Example: Lifetime of a bulb is expected to be 10,000 hours, estimated with exponential distribution. What is the probability that the bulb will fail in the first 3,000 hours?

$$\lambda = 1/10^5$$

$$P(X < 3,000) = 1 - e^{-\lambda x} = 1 - e^{-10^{-5}*3*10^3} = 0.0296$$

Normal Distribution

It is the most popular continuous distribution with uniform distribution and it has many applications. Also several discrete and continuous distributions (i.e. Binomial, t and chi-squared) converge to normal distribution when data size increases. It is also called Gaussian distribution. Many miscalculations or failed prediction happen because people approximate empirical distributions to normal distribution. It has two main parameters mean (location) μ and standard deviation (scale) σ.
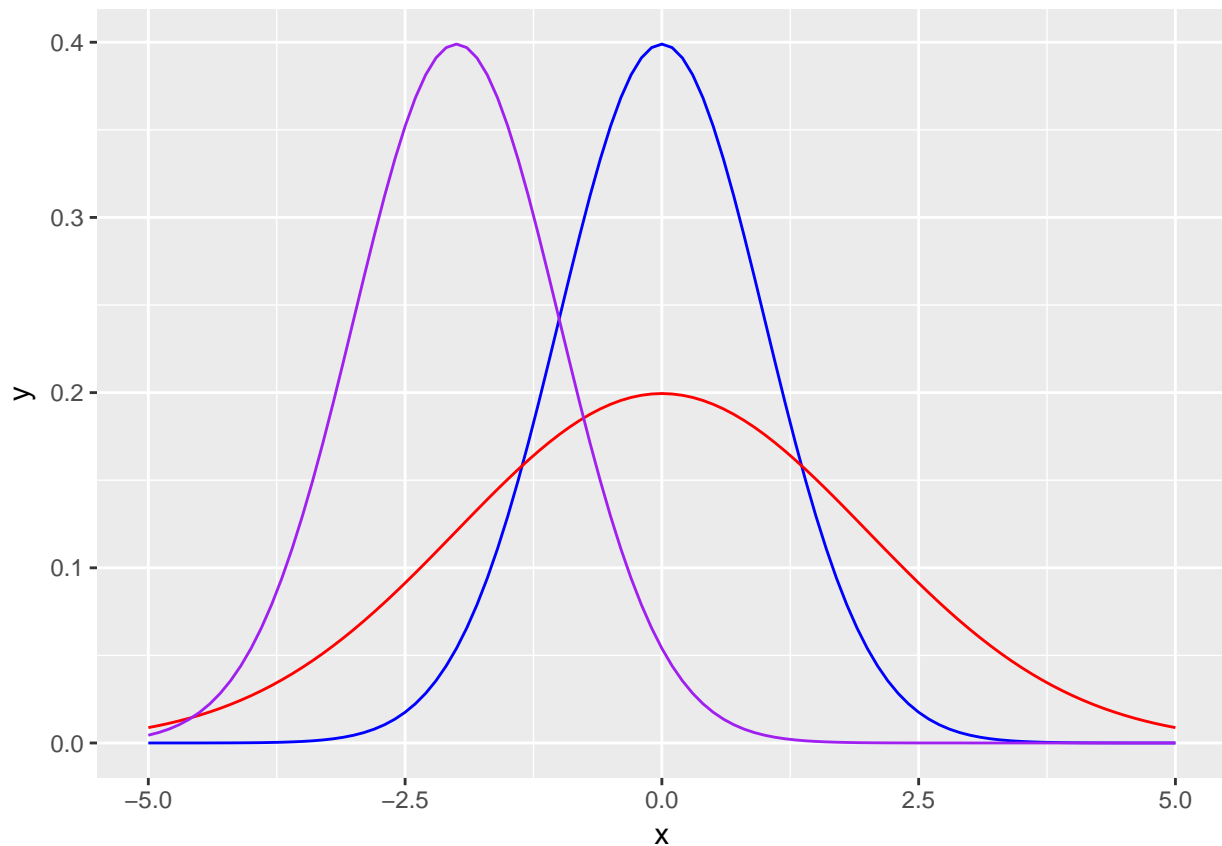
$$X \sim N(\mu, \sigma)$$

- Density: $f(X) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

- $E[X] = \mu$

- $V(X) = \sigma^2$

If $X \sim N(0,1)$, it is called standard normal distribution.

```r
the_seq <- seq(-5,5,length.out=1000)

ggplot(data=data.frame(x=the_seq),aes(x=x)) +
stat_function(fun=dnorm,color="blue") +
stat_function(fun=dnorm,args=list(sd=2),color="red") +
stat_function(fun=dnorm,args=list(mean=-2),color="purple")
```


Example: In a population height of the individuals are normally distributed with mean 170cm and standard deviation 5cm. What is the probability of a randomly selected person has height of 160 cm.

Solution: $P(X < 160; \mu = 170, \sigma = 5) = P(X < \dfrac{160 - 170}{5}) = \phi(-2) = 0.0228$.

Gamma Distribution

First, let's define the gamma function.

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

By extensions there are some interesting properties.

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

If n is a positive integer then

$$\Gamma(n) = (n-1)!$$

.

You can use Gamma Distribution in reliability calculations with multiple components.

- Density: $f(X) = \dfrac{\lambda}{\Gamma(r)}(\lambda x)^{r-1}e^{-\lambda x}, x > 0$.

- $E[X] = r/\lambda$

- $V(X) = r/\lambda^2$

Exponential distribution is a special case of Gamma distribution with $r = 1$.

Weibull Distribution

Some application of Weibull distribution are to estimate the time for failure in multi component electrical or mechanical systems, and modelling wind speed.

- Density: $f(X) = \dfrac{\beta}{\delta}\left(\dfrac{x-\gamma}{\delta}\right)^{\beta-1} e^{-\left(\frac{x-\gamma}{\delta}\right)^{\beta}}$.

- CDF: $F(X) = 1 - e^{-\left(\frac{x-\gamma}{\delta}\right)^{\beta}}$

- $E[X] = \gamma + \delta\Gamma(1 + 1/\beta)$

- $V(X) = \delta^2\left[\Gamma\left(1 + \dfrac{2}{\beta}\right) - \left[\Gamma\left(1 + \dfrac{1}{\beta}\right)\right]^2\right]$

R Functions

R has predefined functions for virtually all distributions. Just write `?Distributions` to the console.

```
dunif #density of uniform distribution
punif #cdf of uniform distribution
qunif #quantile of uniform distribution
runif #uniform distributed random variate

dnorm #normal distribution
dgamma #gamma distribution
dweibull #weibull distribution
dexp #exponential distribution
```