IE-231 In-Class Activity - Week 10 Solutions

Nov 28, 2017

1. Suppose people arrive at a bank with poisson rate $\lambda = 4$ per hour.
   a) What is the probability that 5 people arrive in the first half hour?
   b) What is the probability that at least 3 people arrive in the first hour?

   Solution

a) $P(X = 5|\lambda t = 4 * 0.5) = \dfrac{e^{-\lambda t}(\lambda t)^5}{5!}$

```
#pdf of poisson
dpois(5,lambda=4*0.5)
```

```
## [1] 0.03608941
```

b) $P(X \geq 3|\lambda t = 4) = 1 - P(X \leq 3|\lambda t = 4) = 1 - \sum_{i=0}^{3} \dfrac{e^{-\lambda t}(\lambda t)^i}{i!}$\$

```
#pdf of poisson
1 - ppois(3,lambda=4)
```

```
## [1] 0.5665299
```

2. Patients arrive at the doctor's office according to Poisson distribution with $\lambda = 4/\text{hour}$.

   a) What is the probability of getting less than or equal to 8 patients within 2 hours?
   b) Suppose each arriving patient has 25% chance to bring a person to accompany. There are 20 seats in the waiting room. At least many hours should pass that there is at least 50% probability that the waiting room is filled with patients and their relatives?

a) $P(X \leq 8|\lambda t = 4 * 2) = \sum_{i=0}^{8} \dfrac{e^{-\lambda t}(\lambda t)^i}{i!}$

```
#cdf of poisson
ppois(8,lambda=4*2)
```

```
## [1] 0.5925473
```

b) First let's define the problem. Define n_p as the number of patients and n_c is the number of company. We want $n_p + n_c \geq 20$ with probability 50% or higher for a given t^*. Or to paraphrase, we want $n_p + n_c \leq 19$ w.p. 50% or lower.

What is n_c affected by? n_p. It is actually a binomial distribution problem. $P(n_c = i|n_p) = \binom{n_p}{i}(0.5)^i * (0.5)^{n_p - i}$. It is even better if we use cdf $P(n_c \leq k|n_p) = \sum_{i=0}^{k} \binom{n_p}{i}(0.5)^i * (0.5)^{n_p - i}$.

We know the arrival of the patients is distributed with poisson. So, $P(n_p = j|\lambda t^*) = \dfrac{e^{-\lambda t}(\lambda t)^j}{j!}$. So $P(j + k \leq N) = \sum_{a=0}^{j} P(n_p = a|\lambda t^*) * P(n_c \leq N - a|n_p = a)$. Remember it is always $n_c \leq n_p$.

```
#Let's define a function
calculate_probability<-function(N=19,t_star=1,lambda=4,company_prob=0.25){
    #N is the max desired number of patients
    the_prob<-0
    for(n_p in 0:N){
        the_prob <- the_prob + dpois(n_p,lambda=lambda*t_star)*pbinom(q=min(N-n_p,n_p),size=n_p,prob
    }

    return(the_prob)
```

```r
}

#Try different t_stars so probability is below 0.5
calculate_probability(t_star=3)
```

```
## [1] 0.8380567
```

```r
calculate_probability(t_star=3.3)
```

```
## [1] 0.7443518
```

```r
calculate_probability(t_star=3.955)
```

```
## [1] 0.49914
```

3. Suppose the the pdf of a random variable x is $f(x) = \dfrac{a}{(1-x)^{1/3}}$ for $0 < x < 1$ and 0 for other values of x. (Note: There was a typo in the in-class exercise saying the interval is $0 < x < 1$)

    a) Find the constant a.
    b) Find cdf of F(X < 3/4).

Integral of $f(x)$ should be 1. So $\int_0^1 f(x)dx = a * (-3/2(1-x)^{2/3})|_0^1 = 3a/2$. Then $a = 2/3$.

Use the same integral $\int_0^{3/4} f(x)dx = -(1-x)^2/3|_0^{3/4} = 1 - (1/4)^{2/3} = 0.603$

4. Let X and Y be the random variables and $f(x, y)$ is the probability density function of the joint distribution. Suppose $f(x, y) = a(\dfrac{5x}{7} + \dfrac{9y^3}{2})$ if $0 < x < 2$ and $-1 < y < 1$ (0 otherwise).

    a) Find a.
    b) Find the marginal distribution of y ($h(y)$) and $h(y < 0.5)$.
    c) Find the conditional distribution of $f(y|x)$.

    a) $\int_{-1}^1 \int_0^2 a(\dfrac{5x}{7} + \dfrac{9y^3}{2})dxdy = \int_{-1}^1 a(\dfrac{5x^2}{14} + \dfrac{9xy^3}{2})dy|_0^2 = \int_{-1}^1 a(\dfrac{10}{7} + 9y^3)dy$. (This is also $h(y)$ if a is known.) $a(\dfrac{10y}{7} + \dfrac{9y^4}{4})|_{-1}^1 = a(20/7)$. In order to be a distribution it should be equal to 1. So $a = 7/20$.

    b) As given in (a) $h(y) = 7/20 * (\dfrac{10}{7} + 9y^3)$. So $h(y < 0.5) = \int_{-1}^{0.5} 7/20 * (\dfrac{10}{7} + 9y^3)dy = 7/20 * (\dfrac{10y}{7} + \dfrac{9y^4}{4})|_{-1}^{0.5} = 0.0335$

    c) We need to find $g(x) = \int_{-1}^1 7/20(\dfrac{5x}{7} + \dfrac{9y^3}{2})dy = x/2$. $f(y|x) = f(x, y)/g(x)$.