Initial Concepts of Probability

IE231 - Lecture Notes - Week 1

Feb 20, 2018

Definitions

- **Probability** is the quantification of event uncertainty. For instance, probability of getting (H)eads in a coin toss is 1/2. Deterministic models will give the same results given the same inputs (e.g. 2 times 2 is 4), but probabilistic models might yield different outcomes.
- An **experiment** is a process that generates data. For instance, tossing a coin is an experiment. **Outcome** is the realization of an experiment. Possible outcomes for a coin toss is Heads and Tails.
- **Sample space** ($\mathbb{S}$) is the collection of all the possible outcomes of an experiment. Sample space of the coin toss is $\mathbb{S} = \{H, T\}$. Sample space of two coin tosses experiment is $\mathbb{S} = \{HH, HT, TH, TT\}$. Sample space can be discrete (i.e. coin tosses) as well as continuous (i.e. All real numbers between 1 and 3. $\mathbb{S} = \{x|1 \leq x \leq 3, x \in \mathbb{R}\}$) *(Side note: Sample space is not always well defined.)*
- An **event** is a subset of sample space. While outcome represents a realization, event is an information. Probability of an event $P(A)$, say getting two Heads in two coin tosses is $P(A) = 1/4$.
- A **random variable** represents an event is dependent on a probabilistic process. On the other hand, a **deterministic variable** is either a constant or a decision variable. For instance, value of the dollar tomorrow can be considered a random variable but the amount I will invest is a decision variable (subject to no probabilistic process) and spot (current) price of the dollar is a constant.

Set Operations

- **Complement** of an event (A') with respect to the sample space represents all elements of the sample space that are not included by the event (A). For instance, complement of event $A = \{HH\}$ is $A' = \{HT, TH, TT\}$
- **Union** of two events A and B ($A \cup B$) is a set of events which contains all elements of the respective events. For example, say A is the set that contains events which double Heads occur ($A = \{HH, HT, TH\}$) and B is the set which Tails occur at least once ($B = \{TT, HT, TH\}$). The union is $A \cup B = \{HH, TH, HT, TT\}$.
- **Intersection** of two events A and B ($A \cap B$) contains the common elements of the events. For example, say A is the set that contains events which Heads occur at least once ($A = \{HH, HT, TH\}$) and B is the set which Tails occur at least once ($B = \{TT, HT, TH\}$). The intersection is $A \cap B = \{TH, HT\}$.
- **Mutually exclusive** or disjoint events mean that two events have empty intersection ($A \cap B = \emptyset$) and their union ($A \cup B$) contains the same amount of elements as the sum of their respective number of elements. Also $P(A \cap B) = 0$ and $P(A \cup B) = P(A) + P(B)$. For example getting double Heads (HH) and double Tails (TT) are mutually exclusive events.

Axioms of Probability

1. Any event A belonging to the sample space $A \in \mathbb{S}$ should have nonnegative probability ($P(A) \geq 0$).
2. Probability of the sample space is one ($P(\mathbb{S}) = 1$).
3. Any disjoint events ($A_i \cap A_j = \emptyset \ \forall_{i,j \in 1 \ldots n}$) satisfies $P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$.

Other Set and Probability Rules

- $(A')' = A$
- $S' = \emptyset$
- $\emptyset' = S$
- $(A \cap B)' = A' \cup B'$
- $(A \cup B)' = A' \cap B'$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
- $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- $(A \cup B) \cup C = A \cup (C \cup B)$
- $(A \cap B) \cap C = A \cap (C \cap B)$
- $A \cup A' = \mathbb{S}$ and $A \cap A' = \emptyset$ so $P(A) = 1 - P(A')$. This is especially useful for many problems. For example the probability of getting at least one Heads in a three coin tosses in a row is $1 - P(\{TTT\}) = 7/8$, the complement of no Heads in a three coin tosses in a row. Otherwise, you should calculate the following expression.

$$P(\{HTT\}) + P(\{THT\}) + P(\{TTH\}) + P(\{HHT\}) + P(\{HTH\}) + P(\{THH\}) + P(\{HHH\}) = 7/8$$

- If $A \subseteq B$ then $P(A) \leq P(B)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

Counting

Counting rules will help us enumerate the sample space. It will include multiplication rule, permutation and combination.

Multiplication Rule

If I have a series of independent events, say 1 to k, and number of possible outcomes are denoted with n_1 to n_k; total number of outcomes in the sample space would be $n_1 n_2 \dots n_k$.

Take a series of coin tosses in a row. If I toss a coin its sample space consists of 2 elements such as $\{H, T\}$. If I toss 2 coins the sample space would be 2*2 $\{HH, HT, TH, TT\}$. If I toss 3 coins, the sample space would be 2*2*2 $\{HHH, HTH, THH, TTH, HHT, HTT, THT, TTT\}$.

A poker card consists of a type and a rank. There are four types of playing cards (clubs, diamonds, hearts and spades) and 13 ranks (A - 2 to 10 - J - Q - K). Number of cards in a deck is 4*13 = 52.

Permutation Rule

Permutation is the arrangement of all or a subset of items.

- Given a set of items, say $A = a, b, c$ in how many different ways I can order the elements? Answer is n!. In our case it is, $3! = 3.2.1 = 6$.

$$A = \{a, b, c\}, \{b, a, c\}, \{b, c, a\}, \{c, a, b\}, \{c, b, a\}, \{a, c, b\}$$

- Suppose there are 10 (n) participants in a competition and 3 (r) medals (gold, silver and bronze). How many possible outcomes are there? Answer is $n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!} = \frac{10!}{(10-3)!} = 720$.

- If there are more than one same type items in a sample, then the permutation becomes $\frac{n!}{n_1!n_2!\ldots n_k!}$ where $\sum n_i = n$.

  For example enumerate the different outcomes of four coin tosses which result in 2 heads and 2 tails. Answer is $\frac{4!}{2!2!} = 6$

$$A = \{HHTT, HTTH, HTHT, THTH, THHT, TTHH\}$$

Combination Rule

Suppose we want to select r items from n items and the order does not matter. So the number of different outcomes can be found using $\binom{n}{r} = \frac{n!}{(n-r)!r!}$.

Out of 10 students how many different groups of 2 students can we generate? Answer $\frac{10!}{8!2!} = 45$