Final Lecture Mini In-Class Solutions

IE231 - Lecture 14 Solutions

Dec 26, 2017

These solutions are provided because of overwhelming student requests. As always, they come with absolutely NO WARRANTY that solutions are correct. You should check them yourself, ask your instructor if something is not right. Also, keep in mind you might be asked more explicit calculations than the provided solutions here. Plus, some typo are corrected in this document.

1. In how many ways can you arrange the letters of "SCIENTISTS"?

   10 letters, 3 vowels, 7 consonants, 3 Ss, 2 Is, 2 Ts.

   a) Any order?

   $$\frac{10!}{3!2!2!}$$

   b) Vowels together?

   Consider all vowels a single letter. $\frac{(7+1)!}{3!2!}\frac{3!}{2!}$

   c) Vowels separate?

   7 consonants, 8 potential spots for vowels. $\frac{7!}{3!2!}\frac{8!}{(8-3)!3!}\frac{3!}{2!}$

2. In a box there are 15 balls, 5 white 10 black. If I randomly pick 6 balls from the box, what is the probability that it will be 3 white and 3 blacks?

   $N = 15$,$n = 10$,$k = 6$,$x = 3$. $\frac{\binom{10}{3}\binom{5}{3}}{\binom{15}{6}}$

3. There are 18 people; 8 women, 10 men. Suppose you want to form a group of 4 people with either at least 1 woman or man. How many ways are there?

   Find all combinations, subtract undesirable (no woman groups, no man groups) groups.

   $\binom{18}{4} - \binom{10}{4} - \binom{8}{4}$

4. In a marmalade shop people buy strawberry wp 0.5, apricot 0.3 and peach 0.2. Those who buy marmalade like the strawberry wp 0.8, apricot 0.7 and peach 0.9.

   X is either (L)ike, (N)ot like. B is buy (S),(A),(P).

   a) What is the probability that a random buyer will like the product he/she bought?

   $P(X = L) = 0.5 * 0.8 + 0.3 * 0.7 + 0.2 * 0.9$

   b) Suppose a buyer did not like her marmalade. What is the probability that the bought apricot marmalade?

   $P(B = A|X = N) = \frac{0.3 * (1 - 0.7)}{1 - P(X = L)}$

5. Ayşe is an actress. She goes to auditions to get roles in movies. She gets an offer from an audition with probability 0.6. She went to 8 auditions last week.

   `Binomial (read as almost trivial).`

a) What is the probability that she got 4 offers?

$P(X=4) = \binom{8}{4}(0.6)^4 * (0.4)^4$

b) What is the probability that she got at least 2 offers?

$P(X \geq 2) = 1 - P(X \leq 1) = 1 - \left[\binom{8}{0}(0.6)^0 * (0.4)^8 + \binom{8}{1}(0.6)^1 * (0.4)^7\right]$

6. The cake shop has three kinds of cakes; chocolate cake, strawberry cake and cheesecake. A customer orders (C)hocolate cake with probability 0.3, (S)trawberry cake 0.55 and c(H)eesecake 0.15.

   a) What is the probability that at least three customers among first 5 customers order chocolate or strawberry cakes?

   $P(X = \{C, S\}) = 0.3 + 0.55 = 0.85$ Define $P(n \geq 3) = \sum_{i=3}^{5} \binom{5}{i}(0.85)^i * (0.15)^{5-i}$

   b) What is the probability that the first cheesecake is ordered by the 3rd customer or before?

   0.15 + 0.85*0.15 + 0.85ˆ2*0.15

7. A machine produces 20 items, 12 of which are non-defective. The items are randomly selected without replacement. The sixth selected item is found to be non-defective. What is the probability that this is the third non-defective one?

   If 6th item is the 3rd non-defective, then in the first 5 items there should be 2 non-defectives in any order.

$$\frac{\binom{8}{3}\binom{12}{2}}{\binom{20}{5}}$$

8. When a Simpsons fan is asked "Which character from Simpsons family is your favorite?", he/she answers Homer with probability 0.3, Bart w.p. 0.2, Lisa w.p. 0.1, Maggie w.p. 0.2 and Marge w.p. 0.2. In a room of 12 Simpsons fans, what will the probability that there are 6 of them favor Homer, 2 Lisa and 4 Bart?

$$\binom{12}{6, 2, 4}(0.3)^6 * (0.1)^2 * (0.2)^4$$

9. A player throws darts to a special circular target consisting of 4 score regions all centered on the same point. First score region has a radius of r and second region has a radius of $2r$, 3rd region $3r$ and 4th region $4r$. If the player scores at the 1st region he gets 50 points, 2nd region 25, 3rd 10 and 4th 5 points. He has an equal chance to throw the dart within the target (assume probability of missing the target is zero). If he throws 10 darts, what is his expected total score?

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Whole target area is $\pi (4r)^2 = 16\pi r^2$. Area of the first region is $\pi r^2$, second regio
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$$E[X] = 50 * 1/16 + 25 * 3/16 + 10 * 5/16 + 5 * 7/16$$

10. A bowling player has the probability 0.8 to score a strike at each shot. He makes a bet with his friends If he makes at least 8 strikes out of 10 he will be given 5 TL, but if he makes fewer than or equal to four strikes he will lose 10 TL.

    a) What is the probability that he gets the money?

    $P(X \geq 8) = \sum_{i=8}^{10} \binom{10}{i}(0.8)^i * (0.2)^{10-i}$

    b) What is the expected earnings of the player?

    $E[X] = 5 * P(X \geq 8) - 10 * P(X \leq 4)$

c) What is the variance of his earnings?

$V(X) = E[X^2] - E[X]^2 = (25 * P(X \geq 8) + 100 * P(X \leq 4)) - (5 * P(X \geq 8) - 10 * P(X \leq 4))^2$

11. People arrive at a concert hall with poisson rate 10 per minute. The concert hall has a capacity of 500 people.

    a) What is the probability that the concert hall is full in an hour?

    $\lambda * t = 10 * 60 = 600$. $P(X \geq 500) = 1 - P(X < 500) = \sum_{i=0}^{499} \dfrac{e^{-\lambda * t}(\lambda * t)^i}{i!}$

    b) The manager wants to know when the concert hall will be completely full. Give her a time with 90% probability of being true.

    Find a t^* that $P(X \geq 500 | \lambda * t^*) = 0.9$.

12. Time between customer arrivals in a cafe is exponential with the mean value of 6 minutes.

    $E[X] = 1/\lambda$ for exponential distribution. Then $\lambda = 1/6 perminute$.

    a) What is the probability that no customers arrive in 15 minutes?

    $P(X = 0 | \lambda * t) = \dfrac{e^{-15/6} * (15/6)^0}{0!} = e^{-15/6}$ (Exactly the same as exponential $P(t > 15)$. What a coincidence.)

    b) What is the interarrival time if the probability of a customer to arrive is 0.9?

    $P(X > t) = 0.1$. $e^{-t/6} = 0.1$, then $t = -6 * log(0.1)$.

    c) What is the probability that 10 customers arrive in the first hour?

    $P(X = 10 | \lambda * t) = \dfrac{e^{-10} * (10)^{10}}{10!}$

    d) What is the probability of getting the first customer in 15 minutes if no customer arrived in the first 10 minutes?

    $P(X = 0 | t = 5)$

13. There are two alarms, each sets off at any random time within 10 and 20 minutes respectively. What is the probability that the second alarm (20 min) sets off later than the first one (10 min)?

    Define X as the first alarm and Y as the second alarm. $P(Y > X) =?$

    $\int_0^{10} \int_x^{20} f(x,y) dy dx = \int_0^{10} \int_x^{20} \dfrac{x-0}{10-0} \dfrac{y-0}{20-0} dy dx = \int_0^{10} \int_x^{20} \dfrac{xy}{200} dy dx$

14. There is a speed radar between the 15th and 45th km of a 60 km road, randomly placed. In order to avoid a traffic penalty, the speed of a car should be below 60 km/h. The car's max speed is 120 km/h. What is the expected time of finishing the road if the driver wants to keep the risk of getting caught under 25% probability? *(Assume you can instantly change the speed of the car, no acceleration or deceleration)*

    Get the first and last 15 kms with 120 km/h. 120/30 = 4h. Go only 25% of the 30 km road with 60 km/h and rest with 120 km/h (you can split it in n pieces, no problem). So 7.5 km with 60km/h and 22.5 km with 120 km/h. Sum the durations.

15. Meeting durations in a company are normally distributed with mean 1 hour and standard deviation 10 minutes.

    Define $\mu = 60$, $\sigma = 10$.

    a) What is the probability that a meeting finishes within 45 minutes?

    $\Phi(\dfrac{45-60}{10}) = \Phi(-1.5) = 0.067$

b) What is the duration of a meeting that could have lasted longer with 5% probability?

$\Phi(\dfrac{x-60}{10}) = 0.95$, then $1.64 * 10 + 60 = 76.4$.

16. A rice package consists of 750 gr of rice. Though the weight of the package varies with standard deviation of 15 gr.

a) What is the probability that the package contains at least 775 grams of rice?

$1 - \Phi(\dfrac{775-750}{15}) = 0.047$

b) What is the range of package weight around the mean 90% of the time?

Since probability is around the mean, tails should contain 5% probability each. You can also call this a confidence interval.

Upper limit $\Phi^{-1}(0.95) = \dfrac{x-750}{15}$. $x = 750 + 1.96 * 15$. Lower limit $y = 750 - 1.96 * 15$.

c) In order to prevent light weighted packages what should be the mean weight so that 90% of the packages weight 750 gr or more?

This question asks to move μ to μ^*.

$1 - \Phi(\dfrac{750-\mu^*}{15}) = 0.9$, find μ^*.

17. Three balls are chosen randomly from an urn consisting of 7 green balls and 4 yellow balls. Let Y_i be the ith pick, it is 1 if green, 0 otherwise.

Left to student.

a) Find the joint probabilities of X_1 and X_2.
b) Find the joint probabilities of X_1, X_2 and X_3.

18. 10% of lightbulbs expire in the first 1000 hours. What is the probability that a lightbulb wouldn't expire in at least 1500 hours? (Assume exponential distribution)

$P(t < 1000) = 0.1 = 1 - e^{-\lambda * t}$, find t and find $P(t > 1500)$.

19. X and Y have the joint pdf of $f(x, y) = x + cy^2$, $0 < x < 1$ and $0 < y < 1$.

Left to student.

a) Find c.
b) Find $P(X \leq 0.5, Y \leq 0.5)$

20. You are playing the three cups game. There is a marble in one of the cups. You pick a cup at random and if you choose the cup with the marble, you win. Each play costs 5 TL and if you win you earn 15 TL. What is your expected earnings if you play until you win?

Conditional expectation. $E[X] = 1/3 * (15 - 5) + 2/3 * ((-5) + E[X])$, $E[X] = 0$.